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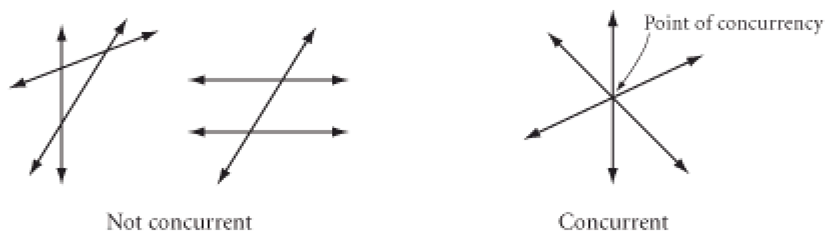
LESSON

3.7 Constructing Points of Concurrency

Nothing in life is to be feared, it is only to be understood.

MARIE CURIE

You now can perform a number of constructions in triangles, including angle bisectors, perpendicular bisectors of the sides, medians, and altitudes. In this lesson and the next lesson you will discover special properties of these lines and segments. When three or more lines have a point in common, they are **concurrent.** Segments, rays, and even planes are concurrent if they intersect in a single point.



The point of intersection is the **point of concurrency.**

Investigation 1
Concurrence

You will need

- patty paper
- geometry software (optional)

In this investigation you will discover that some special lines in a triangle have points of concurrency.

As a group, you should investigate each set of lines on an acute triangle, an obtuse triangle, and a right triangle to be sure that your conjectures apply to all triangles.

Step 1 Draw a large triangle on patty paper. Make sure you have at least one acute triangle, one obtuse triangle, and one right triangle in your group.

Step 2 Construct the three angle bisectors for each triangle. Are they concurrent?

Compare your results with the results of others. State your observations as a conjecture.

Angle Bisector Concurrency Conjecture C-9

The three angle bisectors of a triangle _?_.

Step 3 Draw a large triangle on a new piece of patty paper. Make sure you have at least one acute triangle, one obtuse triangle, and one right triangle in your group.

Step 4 Construct the perpendicular bisector for each side of the triangle and complete the conjecture.

Perpendicular Bisector Concurrency Conjecture C-10

The three perpendicular bisectors of a triangle __?__.

Step 5 Draw a large triangle on a new piece of patty paper. Make sure you have at least one acute triangle, one obtuse triangle, and one right triangle in your group.

Step 6 Construct the lines containing the altitudes of your triangle and complete the conjecture.

Altitude Concurrency Conjecture C-11

The three altitudes (or the lines containing the altitudes) of a triangle __?__.

Step 7 For what kind of triangle will the points of concurrency be the same point?

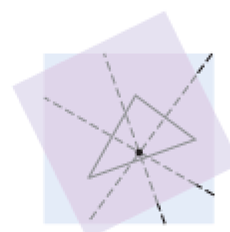
The point of concurrency for the three angle bisectors is the **incenter.** The point of concurrency for the perpendicular bisectors is the **circumcenter.** The point of concurrency for the three altitudes is called the **orthocenter.** Use these definitions to label each patty paper from the previous investigation with the correct name for each point of concurrency. You will investigate a triangle's medians in the next lesson.

Investigation 2
Circumcenter

You will need

- construction tools
- geometry software (optional)

In this investigation you will discover special properties of the circumcenter.



Step 1 Using your patty paper from Steps 3 and 4 of the previous investigation, measure and compare the distances from the circumcenter to each of the three vertices. Are they the same? Compare the distances from the circumcenter to each of the three sides. Are they the same?

Step 2 Tape or glue your patty paper firmly on a piece of regular paper. Use a compass to construct a circle with the circumcenter as the center and that passes through any one of the triangle's vertices. What do you notice?

Step 3 Use your observations to state your next conjecture.

Circumcenter Conjecture C-12

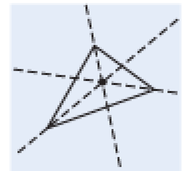
The circumcenter of a triangle __?__.

Investigation 3
Incenter

You will need

- construction tools
- geometry software (optional)

In this investigation you will discover special properties of the incenter.



Step 1 Using the patty paper from the first two steps of Investigation 1, measure and compare the distances from the incenter to each of the three sides. (Remember to use the perpendicular distance.) Are they the same?

Step 2 Construct the perpendicular from the incenter to any one of the sides of the triangle. Mark the point of intersection between the perpendicular line and the side of the triangle.

Step 3 Tape or glue your patty paper firmly on a piece of regular paper. Use a compass to construct a circle with the incenter as the center and that passes through the point of intersection in Step 2. What do you notice?

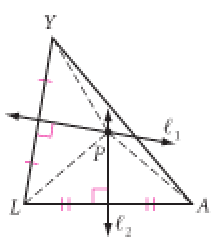
Step 4 Use your observations to state your next conjecture.

Incenter Conjecture C-13

The incenter of a triangle _?_ .

You just discovered a very useful property of the circumcenter and a very useful property of the incenter. You will see some applications of these properties in the exercises. With earlier conjectures and logical reasoning, you can explain why your conjectures are true.

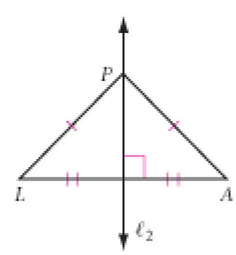
Deductive Argument for the Circumcenter Conjecture



Because the circumcenter is constructed from perpendicular bisectors, the diagram of $\triangle LYA$ at left shows two (of the three) perpendicular bisectors, ℓ_1 and ℓ_2. We want to show that the circumcenter, point P, is equidistant from all three vertices. In other words, we want to show that

$$\overline{PL} \cong \overline{PA} \cong \overline{PY}$$

A useful reasoning strategy is to break the problem into parts. In this case, we might first think about explaining why $\overline{PL} \cong \overline{PA}$. To do that, let's simplify the diagram by looking at just the bottom triangle formed by points P, L, and A.

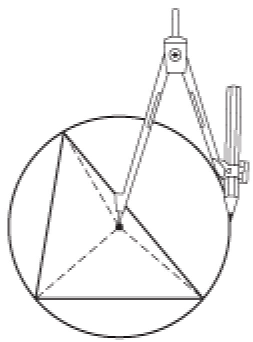


If a point is on the perpendicular bisector of a segment, it is equidistant from the endpoints.

Point P lies on the perpendicular bisector of $\overline{LA}$

$$PA = PL$$

As part of the strategy of concentrating on just part of the problem, think about explaining why $\overline{PA} \cong \overline{PY}$. Focus on the triangle on the left side of $\triangle LYA$ formed by points P, L, and Y.

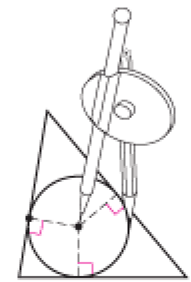


Point P also lies on the perpendicular bisector of $\overline{LY}$

$PL = PY$

Therefore P is equidistant from all three vertices.

$PA = PL = PY$ ■

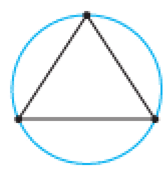


As you discovered in Investigation 2, the circumcenter is the center of a circle that passes through the three vertices of a triangle.

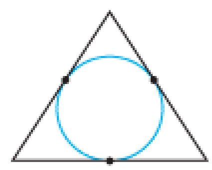
As you found in Investigation 3, the incenter is the center of a circle that touches each side of the triangle. Here are a few vocabulary terms that help describe these geometric situations.

A circle is **circumscribed** about a polygon if and only if it passes through each vertex of the polygon. (The polygon is inscribed in the circle.)

A circle is **inscribed** in a polygon if and only if it touches each side of the polygon at exactly one point. (The polygon is circumscribed about the circle.)



Circumscribed circle
(inscribed triangle)

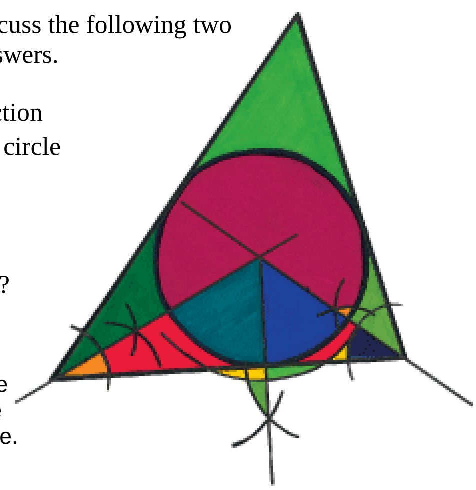


Inscribed circle
(circumscribed triangle)

Developing Proof In your groups discuss the following two questions and then write down your answers.

1. Why does the circumcenter construction guarantee that it is the center of the circle that circumscribes the triangle?

2. Why does the incenter construction guarantee that it is the center of the circle that is inscribed in the triangle?



This geometric art by geometry student Ryan Garvin shows the construction of the incenter, its perpendicular distance to one side of the triangle, and the inscribed circle.

EXERCISES

You will need

Construction tools for Exercises 6, 7, 12–15, and 17

Geometry software for Exercises 10, 11, and 18

For Exercises 1–4, make a sketch and explain how to find the answer.

1. The first-aid center of Mt. Thermopolis State Park needs to be at a point that is equidistant from three bike paths that intersect to form a triangle. Locate this point so that in an emergency, medical personnel will be able to get to any one of the paths by the shortest route possible. Which point of concurrency is it?

Art
CONNECTION

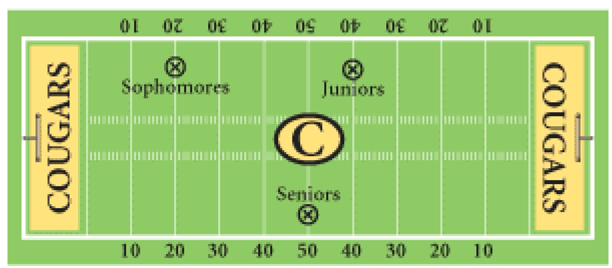
Artist Andres Amador (American, b 1971) creates complex large-scale geometric designs in the sand in San Francisco, California, using construction tools. Can you replicate this design, called *Balance,* using only a compass? For more information about Amador's art, see the links at www.keymath.com/DG .

2. An artist wishes to circumscribe a circle about a triangle in his latest abstract design. Which point of concurrency does he need to locate?

3. Rosita wants to install a circular sink in her new triangular countertop. She wants to choose the largest sink that will fit. Which point of concurrency must she locate? Explain.

4. Julian Chive wishes to center a butcher-block table at a location equidistant from the refrigerator, stove, and sink. Which point of concurrency does Julian need to locate?

5. One event at this year's Battle of the Classes will be a pie-eating contest between the sophomores, juniors, and seniors. Five members of each class will be positioned on the football field at the points indicated at right. At the whistle, one student from each class will run to the pie table, eat exactly one pie, and run back to his or her group. The next student will then repeat the process. The first class to eat five pies and return to home base will be the winner of the pie-eating contest. Where should the pie table be located so that it will be a fair contest? Describe how the contest planners should find that point.



6. ***Construction*** Draw a large triangle. Construct a circle inscribed in the triangle.

7. ***Construction*** Draw a triangle. Construct a circle circumscribed about the triangle.

8. Is the inscribed circle the greatest circle to fit within a given triangle? Explain. If you think not, give a counterexample.

9. Does the circumscribed circle create the smallest circular region that contains a given triangle? Explain. If you think not, give a counterexample.

For Exercises 10 and 11, you can use the **Dynamic Geometry Exploration** Triangle Centers at www.keymath.com/DG .

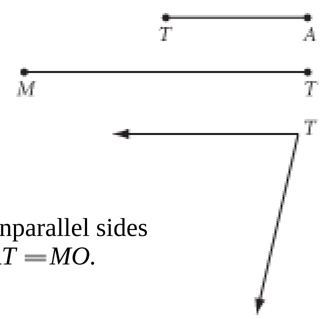
keymath.com/DG

10. Use geometry software to construct the circumcenter of a triangle. Drag a vertex to observe how the location of the circumcenter changes as the triangle changes from acute to obtuse. What do you notice? Where is the circumcenter located for a right triangle?

11. Use geometry software to construct the orthocenter of a triangle. Drag a vertex to observe how the location of the orthocenter changes as the triangle changes from acute to obtuse. What do you notice? Where is the orthocenter located for a right triangle?

Review

Construction Use the segments and angle at right to construct each figure in Exercises 12–15.



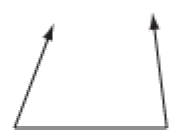
12. ***Mini-Investigation*** Construct $\triangle MAT$. Construct H the midpoint of $\overline{MT}$ and S the midpoint of $\overline{AT}$. Construct the midsegment $\overline{HS}$. Compare the lengths of $\overline{HS}$ and $\overline{MA}$. Notice anything special?

13. ***Mini-Investigation*** An **isosceles trapezoid** is a trapezoid with the nonparallel sides congruent. Construct isosceles trapezoid $MOAT$ with $\overline{MT} \parallel \overline{OA}$ and $AT = MO$. Use patty paper to compare $\angle T$ and $\angle M$. Notice anything special?

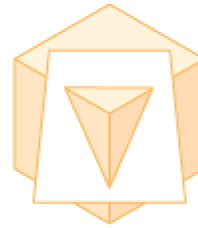
14. ***Mini-Investigation*** Construct a circle with diameter MT. Construct chord $\overline{TA}$. Construct chord $\overline{MA}$ to form $\triangle MTA$. What is the measure of $\angle A$? Notice anything special?

15. ***Mini-Investigation*** Construct a rhombus with TA as the length of a side and $\angle T$ as one of the acute angles. Construct the two diagonals. Notice anything special?

16. Sketch the locus of points on the coordinate plane in which the sum of the x-coordinate and the y-coordinate is 9. (h)



17. ***Construction*** Bisect the missing angle of this triangle. How can you do it without re-creating the third angle? (h)



18. ***Technology*** Is it possible for the midpoints of the three altitudes of a triangle to be collinear? Investigate by using geometry software. Write a paragraph describing your findings.

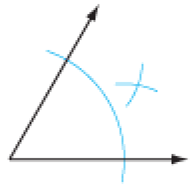
19. Sketch the section formed when the plane slices the cube as shown.

20. Use your geometry tools to draw rhombus $RHOM$ so that $HO = 6.0$ cm and $m\angle R = 120°$.

21. Use your geometry tools to draw kite $KYTE$ so that $KY = YT = 4.8$ cm, diagonal $YE = 6.4$ cm, and $m\angle Y = 80°$. (h)

For Exercises 22–26, complete each geometric construction and name it.

22.



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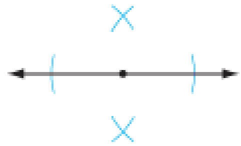

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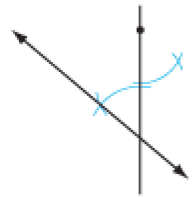

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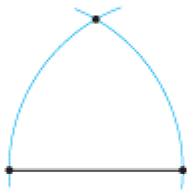

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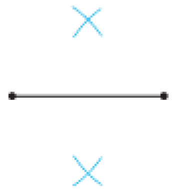

IMPROVING YOUR VISUAL THINKING SKILLS

The Puzzle Lock

This mysterious pattern is a lock that must be solved like a puzzle. Here are the rules:

- You must make eight moves in the proper sequence.
- To make each move (except the last), you place a gold coin onto an empty circle, then slide it along a diagonal to another empty circle.
- You must place the first coin onto circle 1, then slide it to either circle 4 or circle 6.
- You must place the last coin onto circle 5.
- You do not slide the last coin.

Solve the puzzle. Copy and complete the table to show your solution.

Coin movements

Coin	Placed on		Slid to
First	1	→	
Second		→	
Third		→	
Fourth		→	
Fifth		→	
Sixth		→	
Seventh		→	
Eighth	5