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LESSON

2.3 Mathematical Modeling

It's amazing what one can do when one doesn't know what one can't do.

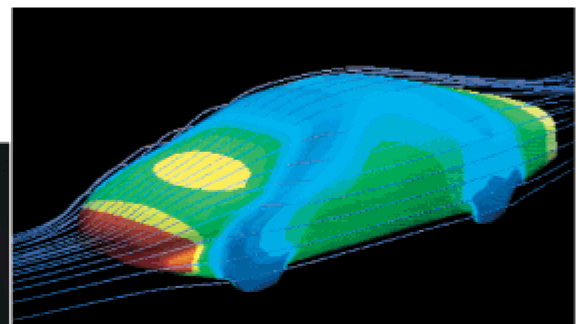
GARFIELD THE CAT

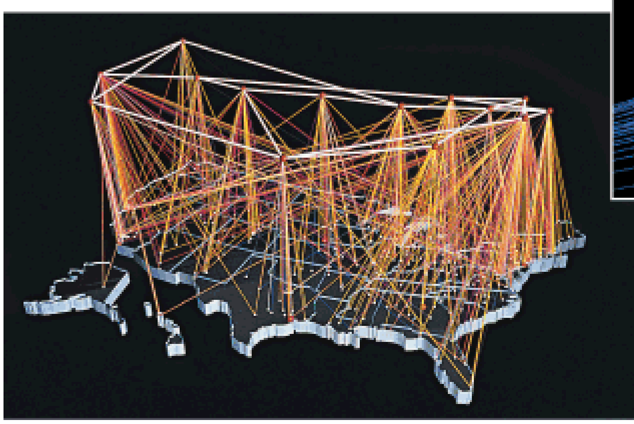
Physical models have many of the same features as the original object or activity they represent, but are often more convenient to study. For example, building a new airplane and testing it is difficult and expensive. But you can analyze a new airplane design by building a model and testing it in a wind tunnel.

In Chapter 1 you learned that geometry ideas such as points, lines, planes, triangles, polygons, and diagonals are **mathematical models** of physical objects.

When you draw graphs or pictures of situations or when you write equations that describe a problem, you are creating mathematical models. A physical model of a complicated telecommunications network, for example, might not be practical, but you can draw a mathematical model of the network using points and lines.

This computer model tests the effectiveness of the car's design for minimizing wind resistance.





This computer-generated model uses points and line segments to show the volume of data traveling to different locations on the National Science Foundation Network.

In this investigation you will attempt to solve a problem first by acting it out, then by creating a mathematical model.

Investigation
Party Handshakes

Each of the 30 people at a party shook hands with everyone else. How many handshakes were there altogether?

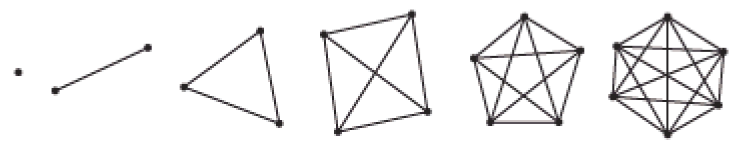
Step 1 Act out this problem with members of your group. Collect data for "parties" of one, two, three, and four people, and record your results in a table.

People	1	2	3	4	...	30
Handshakes	0	1			...	

Step 2 Look for a pattern. Can you generalize from your pattern to find the 30th term?

Acting out a problem is a powerful problem-solving strategy that can give you important insight into a solution. Were you able to make a generalization from just four terms? If so, how confident are you of your generalization? To collect more data, you can ask more classmates to join your group. You can see, however, that acting out a problem sometimes has its practical limitations. That's when you can use mathematical models.

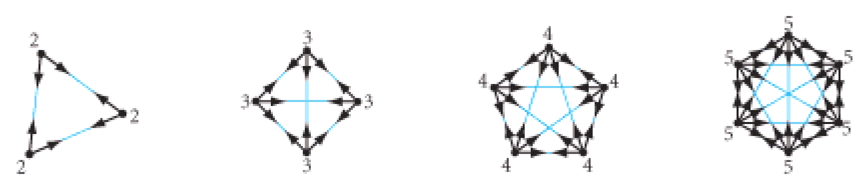
Step 3 Model the problem by using points to represent people and line segments connecting the points to represent handshakes.



Record your results in a table like this one:

Number of points (people)	1	2	3	4	5	6	...	n	...	30
Number of segments (handshakes)	0	1					...		...	

Notice that the pattern does not have a constant difference. That is, the rule is not a linear function. So we need to look for a different kind of rule.



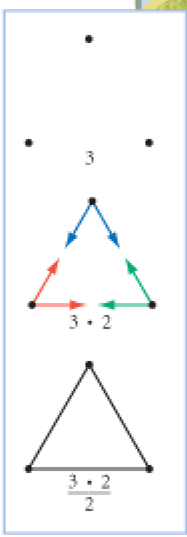
3 points
2 segments per vertex

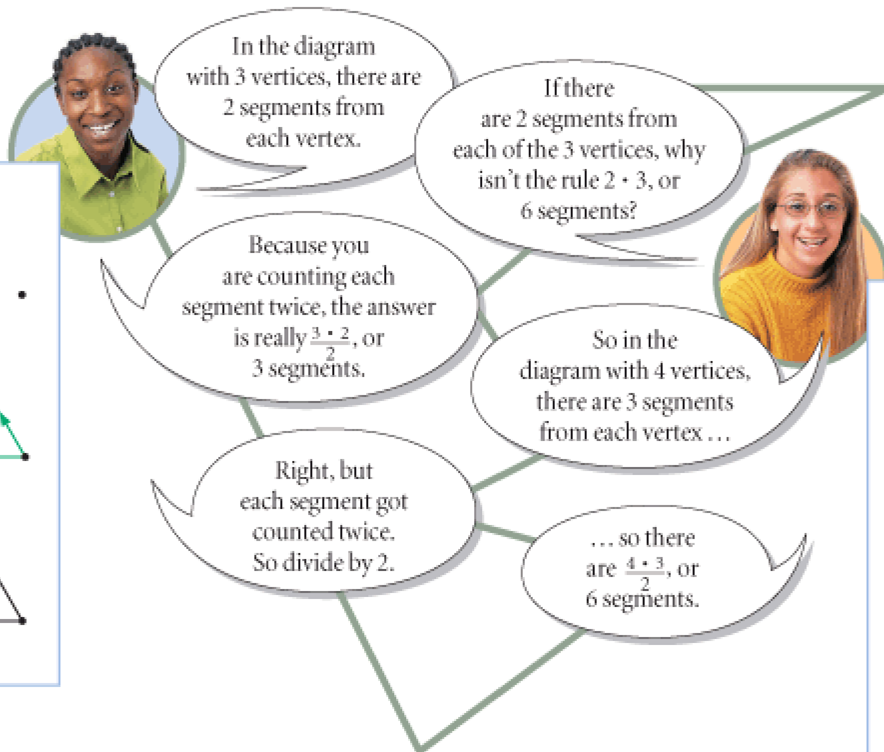
4 points
3 segments per vertex

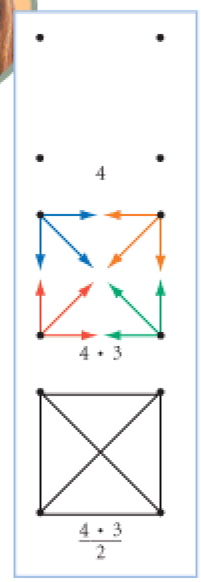
5 points
_?_ segments per vertex

6 points
_?_ segments per vertex

Step 4 Refer to the table you made for Step 3. The pattern of differences is increasing by one: 1, 2, 3, 4, 5, 6, 7. Read the dialogue between Erin and Stephanie as they attempt to use logical reasoning to find the rule.







Let's continue with Stephanie and Erin's line of reasoning.

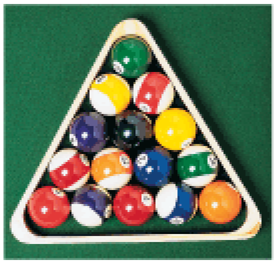
Step 5 In the diagram with 5 vertices, how many segments are there from each vertex? So the total number of segments written in factored form is $\frac{5 \cdot ?}{2}$.

Step 6 Complete the table below by expressing the total number of segments in factored form.

Number of points (people)	1	2	3	4	5	6	. . .	n
Number of segments (handshakes)	$\frac{(1)(0)}{2}$	$\frac{(2)(1)}{2}$	$\frac{(3)(2)}{2}$	$\frac{(4)(3)}{2}$	$\frac{(5)(?)}{2}$	$\frac{(6)(?)}{2}$	. . .	$\frac{(?)(?)}{2}$

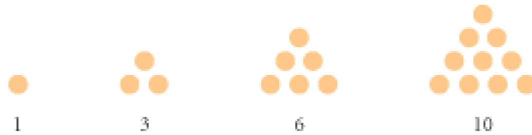
Step 7 The larger of the two factors in the numerator represents the number of points. What does the smaller of the two numbers in the numerator represent? Why do we divide by 2?

Step 8 Write a function rule. How many handshakes were there at the party?



Fifteen pool balls can be arranged in a triangle, so 15 is a triangular number.

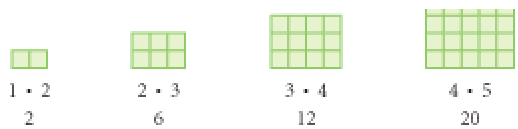
The numbers in the pattern in the previous investigation are called the **triangular numbers** because you can arrange them into a triangular pattern of dots.



The triangular numbers appear in many geometric situations, as you will see in the exercises. They are related to this sequence of **rectangular numbers:**

2, 6, 12, 20, 30, 42, . . .

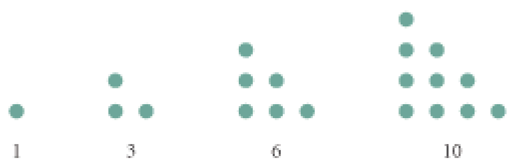
Rectangular numbers can be visualized as rectangular arrangements of objects, in which the length and width are factors of the numbers.



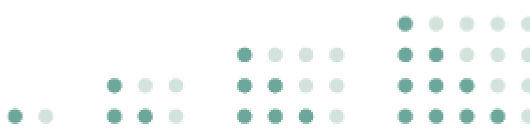
In this sequence, the width is equal to the term number and the length is one more than the term number. The rectangle representing the 3rd term, for instance, has width 3 and length 3 + 1, or 4, so the total number of squares is equal to $3 \cdot 4$, or 12. You can apply this pattern to find any term in the sequence. The 25th rectangle, for example, would have width 25, length 26, and a total number of squares equal to $25 \cdot 26$, or 650.

In general, the nth rectangle in this sequence has a width equal to the term number, n, and a length equal to one more than the term number, or $n + 1$. So the nth rectangular number is $n(n+1)$.

Here is a visual approach to arrive at the rule for the party handshakes problem. If we arrange the triangular numbers in stacks,



you can see that each is half of a rectangular number.

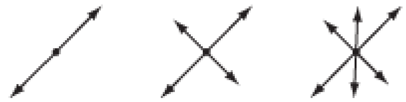


So the triangular array has $\frac{n(n+1)}{2}$ dots.

EXERCISES

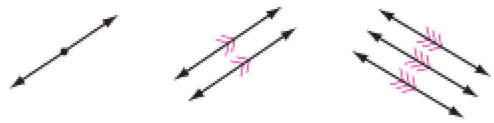
For Exercises 1–6, draw the next figure. Complete a table and find the function rule. Then find the 35th term.

1. Lines passing through the same point are **concurrent.** Into how many regions do 35 concurrent lines divide the plane?

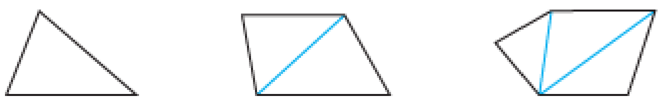


Lines	1	2	3	4	5	...	n	...	35
Regions	2					...		...	

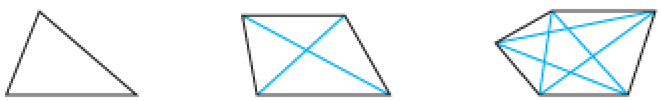
2. Into how many regions do 35 parallel lines in a plane divide that plane?



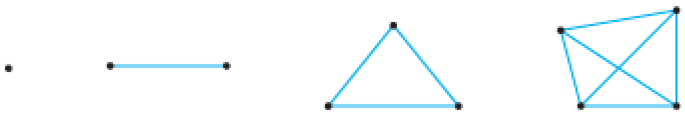
3. How many diagonals can you draw from one vertex in a polygon with 35 sides?



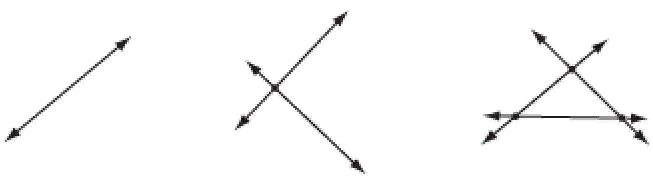
4. What's the total number of diagonals in a 35-sided polygon? (h)



5. If you place 35 points on a piece of paper so that no three points are in a line, how many line segments are necessary to connect each point to all the others? (h)



6. If you draw 35 lines on a piece of paper so that no two lines are parallel to each other and no three lines are concurrent, how many times will they intersect? (h)



7. Look at the formulas you found in Exercises 4–6. Describe how the formulas are related. Then explain how the three problems are related geometrically.

For Exercises 8–10, draw a diagram, find the appropriate geometric model, and solve.

8. If 40 houses in a community all need direct lines to one another in order to have telephone service, how many lines are necessary? Is that practical? Sketch and describe two models: first, model the situation in which direct lines connect every house to every other house and, second, model a more practical alternative.

9. If each team in a ten-team league plays each of the other teams four times in a season, how many league games are played during one season? What geometric figures can you use to model teams and games played?

10. Each person at a party shook hands with everyone else exactly once. There were 66 handshakes. How many people were at the party?

Review

For Exercises 11–19, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.

11. The largest chord of a circle is a diameter of the circle.

12. The vertex of $\angle TOP$ is point O.

13. An isosceles right triangle is a triangle with an angle measuring $90°$ and no two sides congruent.

14. If $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ in point E, then $\angle AED$ and $\angle BED$ form a linear pair of angles.

15. If two lines lie in the same plane and are perpendicular to the same line, they are perpendicular.

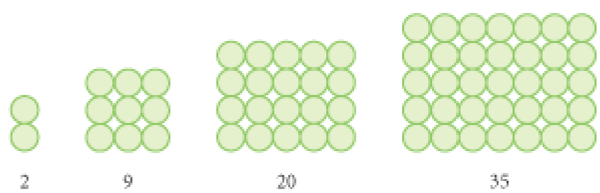
16. The opposite sides of a kite are never parallel.

17. A rectangle is a parallelogram with all sides congruent.

18. A line segment that connects any two vertices in a polygon is called a diagonal.

19. To show that two lines are parallel, you mark them with the same number of arrowheads.

20. The sequence 2, 9, 20, 35, . . . is another example of a rectangular number pattern, as illustrated in the art below. What is the 50th term of this sequence?



IMPROVING YOUR VISUAL THINKING SKILLS

Pentominoes II

In Pentominoes I, you found the 12 pentominoes. Which of the 12 pentominoes can you cut along the edges and fold into a box without a lid? Here is an example.